## 4.5 Angular momentum

### Orbital angular momentum

| | | | |
|---|---|---|---|
| Angular momentum operators | $\hat{\boldsymbol{L}} = \boldsymbol{r} \times \hat{\boldsymbol{p}}$ | (4.101) | $\boldsymbol{L}$ angular momentum |
| | $\hat{L}_z = \frac{\hbar}{\mathrm{i}}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ | (4.102) | $\boldsymbol{p}$ linear momentum |
| | $= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \phi}$ | (4.103) | $\boldsymbol{r}$ position vector |
| | $\hat{L}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2$ | (4.104) | $xyz$ Cartesian coordinates |
| | $= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ | (4.105) | $r\theta\phi$ spherical polar coordinates; $\hbar$ (Planck constant)/$(2\pi)$ |
| Ladder operators | $\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y$ | (4.106) | $\hat{L}_\pm$ ladder operators |
| | $= \hbar e^{\pm\mathrm{i}\phi}\left(\mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right)$ | (4.107) | $Y_l^{m_l}$ spherical harmonics |
| | $\hat{L}_\pm Y_l^{m_l} = \hbar[l(l+1) - m_l(m_l \pm 1)]^{1/2} Y_l^{m_l \pm 1}$ | (4.108) | $l, m_l$ integers |
| Eigen-functions and eigenvalues | $\hat{L}^2 Y_l^{m_l} = l(l+1)\hbar^2 Y_l^{m_l} \qquad (l \geq 0)$ | (4.109) | |
| | $\hat{L}_z Y_l^{m_l} = m_l \hbar Y_l^{m_l} \qquad (\lvert m_l \rvert \leq l)$ | (4.110) | |
| | $\hat{L}_z[\hat{L}_\pm Y_l^{m_l}(\theta,\phi)] = (m_l \pm 1)\hbar \hat{L}_\pm Y_l^{m_l}(\theta,\phi)$ | (4.111) | |
| | $l$-multiplicity $= (2l+1)$ | (4.112) | |

### Angular momentum commutation relations[a]

| | | | |
|---|---|---|---|
| Conservation of angular momentum[b] | $[\hat{H}, \hat{L}_z] = 0$ | (4.113) | $L$ angular momentum; $p$ momentum; $H$ Hamiltonian; $\hat{L}_\pm$ ladder operators |

| | | | |
|---|---|---|---|
| $[\hat{L}_z, x] = \mathrm{i}\hbar y$ | (4.114) | $[\hat{L}_x, \hat{L}_y] = \mathrm{i}\hbar \hat{L}_z$ | (4.120) |
| $[\hat{L}_z, y] = -\mathrm{i}\hbar x$ | (4.115) | $[\hat{L}_z, \hat{L}_x] = \mathrm{i}\hbar \hat{L}_y$ | (4.121) |
| $[\hat{L}_z, z] = 0$ | (4.116) | $[\hat{L}_y, \hat{L}_z] = \mathrm{i}\hbar \hat{L}_x$ | (4.122) |
| $[\hat{L}_z, \hat{p}_x] = \mathrm{i}\hbar \hat{p}_y$ | (4.117) | $[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+$ | (4.123) |
| $[\hat{L}_z, \hat{p}_y] = -\mathrm{i}\hbar \hat{p}_x$ | (4.118) | $[\hat{L}_-, \hat{L}_z] = \hbar \hat{L}_-$ | (4.124) |
| $[\hat{L}_z, \hat{p}_z] = 0$ | (4.119) | $[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$ | (4.125) |
| | | $[\hat{L}^2, \hat{L}_\pm] = 0$ | (4.126) |

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0 \qquad (4.127)$$

[a]The commutation of $a$ and $b$ is defined as $[a,b] = ab - ba$ (see page 26). Similar expressions hold for $S$ and $J$.
[b]For motion under a central force.

## Clebsch–Gordan coefficients[a]

$$\langle j,-m_j|l_1,-m_1;l_2,-m_2\rangle=(-1)^{l_1+l_2-j}\langle j,m_j|l_1,m_1;l_2,m_2\rangle$$

| $l_1\times l_2$ | | $m_j$: $j$ | $j$ | … |
|---|---|---|---|---|
| $m_1$ | $m_2$ | coefficients | | |
| $m_1$ | $m_2$ | $\langle j,m_j\|l_1,m_1;l_2,m_2\rangle$ | | |
| ⋮ | ⋮ | ⋮ | | |

### $1/2\times1/2$

| $m_1$ | $m_2$ | $m_j=+1$: 1 |
|---|---|---|
| +1/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | $m_j=0$: 1 | 0 |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

### $1\times1/2$

| $m_1$ | $m_2$ | $m_j=+3/2$: 3/2 |
|---|---|---|
| +1 | +1/2 | 1 |

| $m_1$ | $m_2$ | $m_j=+1/2$: 3/2 | 1/2 |
|---|---|---|---|
| +1 | −1/2 | 1/3 | 2/3 |
| 0 | +1/2 | 2/3 | −1/3 |

### $3/2\times1/2$

| $m_1$ | $m_2$ | $m_j=+2$: 2 |
|---|---|---|
| +3/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | $m_j=+1$: 2 | 1 |
|---|---|---|---|
| +3/2 | −1/2 | 1/4 | 3/4 |
| +1/2 | +1/2 | 3/4 | −1/4 |

| $m_1$ | $m_2$ | $m_j=0$: 2 | 1 |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

### $2\times1/2$

| $m_1$ | $m_2$ | $m_j=+5/2$: 5/2 |
|---|---|---|
| +2 | +1/2 | 1 |

| $m_1$ | $m_2$ | $m_j=+3/2$: 5/2 | 3/2 |
|---|---|---|---|
| +2 | −1/2 | 1/5 | 4/5 |
| +1 | +1/2 | 4/5 | −1/5 |

| $m_1$ | $m_2$ | $m_j=+1/2$: 5/2 | 3/2 |
|---|---|---|---|
| +1 | −1/2 | 2/5 | 3/5 |
| 0 | +1/2 | 3/5 | −2/5 |

### $1\times1$

| $m_1$ | $m_2$ | $m_j=+2$: 2 |
|---|---|---|
| +1 | +1 | 1 |

| $m_1$ | $m_2$ | $m_j=+1$: 2 | 1 |
|---|---|---|---|
| +1 | 0 | 1/2 | 1/2 |
| 0 | +1 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | $m_j=0$: 2 | 1 | 0 |
|---|---|---|---|---|
| +1 | −1 | 1/6 | 1/2 | 1/3 |
| 0 | 0 | 2/3 | 0 | −1/3 |
| −1 | +1 | 1/6 | −1/2 | 1/3 |

### $3/2\times1$

| $m_1$ | $m_2$ | $m_j=+5/2$: 5/2 |
|---|---|---|
| +3/2 | +1 | 1 |

| $m_1$ | $m_2$ | $m_j=+3/2$: 5/2 | 3/2 |
|---|---|---|---|
| +3/2 | 0 | 2/5 | 3/5 |
| +1/2 | +1 | 3/5 | −2/5 |

| $m_1$ | $m_2$ | $m_j=+1/2$: 5/2 | 3/2 | 1/2 |
|---|---|---|---|---|
| +3/2 | −1 | 1/10 | 2/5 | 1/2 |
| 1/2 | 0 | 3/5 | 1/15 | −1/3 |
| −1/2 | +1 | 3/10 | −8/15 | 1/6 |

### $2\times1$

| $m_1$ | $m_2$ | $m_j=+3$: 3 |
|---|---|---|
| +2 | +1 | 1 |

| $m_1$ | $m_2$ | $m_j=+2$: 3 | 2 |
|---|---|---|---|
| +2 | 0 | 1/3 | 2/3 |
| +1 | +1 | 2/3 | −1/3 |

| $m_1$ | $m_2$ | $m_j=+1$: 3 | 2 | 1 |
|---|---|---|---|---|
| +2 | −1 | 1/15 | 1/3 | 3/5 |
| +1 | 0 | 8/15 | 1/6 | −3/10 |
| 0 | +1 | 6/15 | −1/2 | 1/10 |

| $m_1$ | $m_2$ | $m_j=0$: 3 | 2 | 1 |
|---|---|---|---|---|
| +1 | −1 | 1/5 | 1/2 | 3/10 |
| 0 | 0 | 3/5 | 0 | −2/5 |
| −1 | +1 | 1/5 | −1/2 | 3/10 |

### $3/2\times3/2$

| $m_1$ | $m_2$ | $m_j=+3$: 3 |
|---|---|---|
| +3/2 | +3/2 | 1 |

| $m_1$ | $m_2$ | $m_j=+2$: 3 | 2 |
|---|---|---|---|
| +3/2 | +1/2 | 1/2 | 1/2 |
| +1/2 | +3/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | $m_j=+1$: 3 | 2 | 1 |
|---|---|---|---|---|
| +3/2 | −1/2 | 1/5 | 1/2 | 3/10 |
| +1/2 | +1/2 | 3/5 | 0 | −2/5 |
| −1/2 | +3/2 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | $m_j=0$: 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|
| +3/2 | −3/2 | 1/20 | 1/4 | 9/20 | 1/4 |
| +1/2 | −1/2 | 9/20 | 1/4 | −1/20 | −1/4 |
| −1/2 | +1/2 | 9/20 | −1/4 | −1/20 | 1/4 |
| −3/2 | +3/2 | 1/20 | −1/4 | 9/20 | −1/4 |

### $2\times3/2$

| $m_1$ | $m_2$ | $m_j=+7/2$: 7/2 |
|---|---|---|
| +2 | +3/2 | 1 |

| $m_1$ | $m_2$ | $m_j=+5/2$: 7/2 | 5/2 |
|---|---|---|---|
| +2 | +1/2 | 3/7 | 4/7 |
| +1 | +3/2 | 4/7 | −3/7 |

| $m_1$ | $m_2$ | $m_j=+3/2$: 7/2 | 5/2 | 3/2 |
|---|---|---|---|---|
| +2 | −1/2 | 1/7 | 16/35 | 2/5 |
| +1 | +1/2 | 4/7 | 1/35 | −2/5 |
| 0 | +3/2 | 2/7 | −18/35 | 1/5 |

| $m_1$ | $m_2$ | $m_j=+1/2$: 7/2 | 5/2 | 3/2 | 1/2 |
|---|---|---|---|---|---|
| +2 | −3/2 | 1/35 | 6/35 | 2/5 | 2/5 |
| +1 | −1/2 | 12/35 | 5/14 | 0 | −3/10 |
| 0 | +1/2 | 18/35 | −3/35 | −1/5 | 1/5 |
| −1 | +3/2 | 4/35 | −27/70 | 2/5 | −1/10 |

### $2\times2$

| $m_1$ | $m_2$ | $m_j=+4$: 4 |
|---|---|---|
| +2 | +2 | 1 |

| $m_1$ | $m_2$ | $m_j=+3$: 4 | 3 |
|---|---|---|---|
| +2 | +1 | 1/2 | 1/2 |
| +1 | +2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | $m_j=+2$: 4 | 3 | 2 |
|---|---|---|---|---|
| +2 | 0 | 3/14 | 1/2 | 2/7 |
| +1 | +1 | 4/7 | 0 | −3/7 |
| 0 | +2 | 3/14 | −1/2 | 2/7 |

| $m_1$ | $m_2$ | $m_j=+1$: 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|
| +2 | −1 | 1/14 | 3/10 | 3/7 | 1/5 |
| +1 | 0 | 3/7 | 1/5 | −1/14 | −3/10 |
| 0 | +1 | 3/7 | −1/5 | −1/14 | 3/10 |
| −1 | +2 | 1/14 | −3/10 | 3/7 | −1/5 |

| $m_1$ | $m_2$ | $m_j=0$: 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|
| +2 | −2 | 1/70 | 1/10 | 2/7 | 2/5 | 1/5 |
| +1 | −1 | 8/35 | 2/5 | 1/14 | −1/10 | −1/5 |
| 0 | 0 | 18/35 | 0 | −2/7 | 0 | 1/5 |
| −1 | +1 | 8/35 | −2/5 | 1/14 | 1/10 | −1/5 |
| −2 | +2 | 1/70 | −1/10 | 2/7 | −2/5 | 1/5 |

[a] Or "Wigner coefficients," using the Condon–Shortley sign convention. Note that a square root is assumed over all coefficient digits, so that "−3/10" corresponds to $-\sqrt{3/10}$. Also for clarity, only values of $m_j \geq 0$ are listed here. The coefficients for $m_j < 0$ can be obtained from the symmetry relation $\langle j,-m_j|l_1,-m_1;l_2,-m_2\rangle = (-1)^{l_1+l_2-j}\langle j,m_j|l_1,m_1;l_2,m_2\rangle$.

## Angular momentum addition[a]

| | | |
|---|---|---|
| Total angular momentum | $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$ (4.128)<br>$\hat{J}_z=\hat{L}_z+\hat{S}_z$ (4.129)<br>$\hat{J}^2=\hat{L}^2+\hat{S}^2+2\widehat{\boldsymbol{L}\cdot\boldsymbol{S}}$ (4.130)<br>$\hat{J}_z\psi_{j,m_j}=m_j\hbar\psi_{j,m_j}$ (4.131)<br>$\hat{J}^2\psi_{j,m_j}=j(j+1)\hbar^2\psi_{j,m_j}$ (4.132)<br>$j$-multiplicity $=(2l+1)(2s+1)$ (4.133) | $\boldsymbol{J},J$ total angular momentum<br>$\boldsymbol{L},L$ orbital angular momentum<br>$\boldsymbol{S},S$ spin angular momentum<br>$\psi$ eigenfunctions<br>$m_j$ magnetic quantum number $\lvert m_j\rvert\le j$<br>$j$ $(l+s)\ge j\ge \lvert l-s\rvert$ |
| Mutually commuting sets | $\{L^2,S^2,J^2,J_z,\boldsymbol{L}\cdot\boldsymbol{S}\}$ (4.134)<br>$\{L^2,S^2,L_z,S_z,J_z\}$ (4.135) | $\{\}$ set of mutually commuting observables |
| Clebsch–Gordan coefficients[b] | $\lvert j,m_j\rangle=\sum_{\substack{m_l,m_s\\ m_s+m_l=m_j}}\langle j,m_j\lvert l,m_l;s,m_s\rangle\lvert l,m_l\rangle\lvert s,m_s\rangle$ (4.136) | $\lvert\cdot\rangle$ eigenstates<br>$\langle\cdot\lvert\cdot\rangle$ Clebsch–Gordan coefficients |

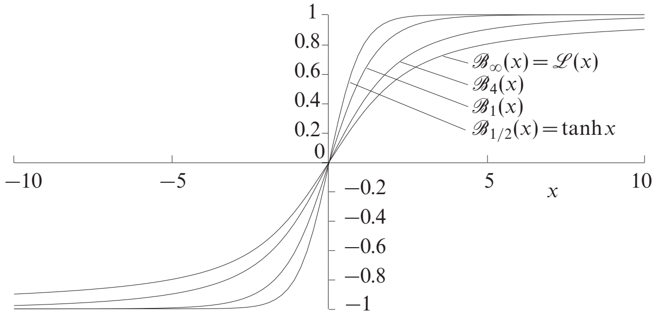
[a]Summing spin and orbital angular momenta as examples, eigenstates $|s,m_s\rangle$ and $|l,m_l\rangle$.
[b]Or "Wigner coefficients." Assuming no $L$–$S$ interaction.

## Magnetic moments

| | | |
|---|---|---|
| Bohr magneton | $\mu_B=\frac{e\hbar}{2m_e}$ (4.137) | $\mu_B$ Bohr magneton<br>$-e$ electronic charge<br>$\hbar$ (Planck constant)/$(2\pi)$<br>$m_e$ electron mass |
| Gyromagnetic ratio[a] | $\gamma=\frac{\text{orbital magnetic moment}}{\text{orbital angular momentum}}$ (4.138) | $\gamma$ gyromagnetic ratio |
| Electron orbital gyromagnetic ratio | $\gamma_e=\frac{-\mu_B}{\hbar}$ (4.139)<br>$=\frac{-e}{2m_e}$ (4.140) | $\gamma_e$ electron gyromagnetic ratio |
| Spin magnetic moment of an electron[b] | $\mu_{e,z}=-g_e\mu_B m_s$ (4.141)<br>$=\pm g_e\gamma_e\frac{\hbar}{2}$ (4.142)<br>$=\pm\frac{g_e e\hbar}{4m_e}$ (4.143) | $\mu_{e,z}$ $z$ component of spin magnetic moment<br>$g_e$ electron $g$-factor ($\simeq 2.002$)<br>$m_s$ spin quantum number ($\pm 1/2$) |
| Landé $g$-factor[c] | $\mu_J=g_J\sqrt{J(J+1)}\mu_B$ (4.144)<br>$\mu_{J,z}=-g_J\mu_B m_J$ (4.145)<br>$g_J=1+\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}$ (4.146) | $\mu_J$ total magnetic moment<br>$\mu_{J,z}$ $z$ component of $\mu_J$<br>$m_J$ magnetic quantum number<br>$J,L,S$ total, orbital, and spin quantum numbers<br>$g_J$ Landé $g$-factor |

[a]Or "magnetogyric ratio."
[b]The electron $g$-factor equals exactly 2 in Dirac theory. The modification $g_e=2+\alpha/\pi+\ldots$, where $\alpha$ is the fine structure constant, comes from quantum electrodynamics.
[c]Relating the spin + orbital angular momenta of an electron to its total magnetic moment, assuming $g_e=2$.

## Quantum paramagnetism




$$\mathscr{B}_J(x) = \frac{2J+1}{2J}\coth\left[\frac{(2J+1)x}{2J}\right] - \frac{1}{2J}\coth\frac{x}{2J} \quad (4.147)$$

**Brillouin function**

$$\mathscr{B}_J(x) \simeq \begin{cases} \frac{J+1}{3J}x & (x \ll 1) \\ \mathscr{L}(x) & (J \gg 1) \end{cases} \quad (4.148)$$

$$\mathscr{B}_{1/2}(x) = \tanh x \quad (4.149)$$

**Mean magnetisation**[a]

$$\langle M \rangle = n\mu_B J g_J \mathscr{B}_J\left(Jg_J\frac{\mu_B B}{kT}\right) \quad (4.150)$$

**$\langle M \rangle$ for isolated spins ($J = 1/2$)**

$$\langle M \rangle_{1/2} = n\mu_B \tanh\left(\frac{\mu_B B}{kT}\right) \quad (4.151)$$

| | |
|---|---|
| $\mathscr{B}_J(x)$ | Brillouin function |
| $J$ | total angular momentum quantum number |
| $\mathscr{L}(x)$ | Langevin function $= \coth x - 1/x$ (see page 144) |
| $\langle M \rangle$ | mean magnetisation |
| $n$ | number density of atoms |
| $g_J$ | Landé $g$-factor |
| $\mu_B$ | Bohr magneton |
| $B$ | magnetic flux density |
| $k$ | Boltzmann constant |
| $T$ | temperature |
| $\langle M \rangle_{1/2}$ | mean magnetisation for $J = 1/2$ (and $g_J = 2$) |

[a] Of an ensemble of atoms in thermal equilibrium at temperature $T$, each with total angular momentum quantum number $J$.